ON A REINFORCED PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN

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1. Introduction

Das (1954) discussed an incomplete randomised block design when the incompleteness is partial but completely balanced. It appears that a similar design when the incompleteness is partially balanced may be of great help for various agricultural, animal husbandry and bio-assay studies. Such a design can actually be called a reinforced P.B.I.B. design and has been discussed in detail in this paper. Such a reinforced P.B.I.B. design may, however, involve sometimes two block sizes, but the method of analysis has been discussed here, assuming (as Das has done for his design) a common intra-block variance for both the block sizes. As in bio-assay and other experiments with litter mates as experimental units, it becomes, sometimes desirable in order to save the wastage of animals, to utilise litters of different sizes in the same experiment, so the design has been developed keeping provision of some extra blocks each having all the treatments. As it is quite likely that the within litter variance is independent of litter sizes, the assumption of one common intra-block variance is justified in such cases. But in agricultural experiments the extra block can better be omitted if the block sizes are quite different.

2. Description

In a randomised complete block design with k treatments and r blocks, let N plots be missing such that

$p =$ Number of affected treatments so that $p < k$;

$q =$ Number of replications missing for each affected treatment;

$s =$ Number of blocks affected such that $r \geqslant s > q$;

$n =$ Number of plots missing in each affected block so that $n < p$;

and with respect to any one of the affected treatments (say θ), the remaining $(p-1)$ affected treatments are devided into m groups of $n_1, n_2 \ldots . n_m$ treatments respectively such that θ is missing in λ_i blocks $(i = 1, 2, \ldots . m)$ with each one of the n_i treatments $(i = 1, 2, \ldots . m)$; and the properties of the parameters of second kind (p_{ij}^k; $i, j, k =$

1, 2,....m) of a P.B.I.B. design [see Bose and Nair (1939)] with $v' = p$, $r' = q$, $b' = s$; $k' = n$; $n_1, n_2, \ldots n_m$; $\lambda_1, \lambda_2, \ldots \lambda_m$; as the parameters of first kind, are also satisfied for the different groupings with respect to any one of the affected treatments;

then

$$pq = sn = N \tag{1}$$

$$\Sigma n_i \lambda_i = q(n-1) \tag{2}$$

$$\sum_j p_{ij}^k = n_i \; (i \neq k)$$

$$= n_i - 1 \; (i = k) \tag{3}$$

$$n_k p_{ij}^k = n_i p_{jk}^i = n_j p_{ik}^j \tag{4}$$

and the design may be called a partially incomplete R.B.D. with partially balanced incompleteness. When $p = k$ and $s = r$, this becomes a P.B.I.B. design with parameters as $v' = k$, $r' = s - q$, $b' = s$, $k' = k - n$; $n_i' = n_i$; $\lambda_i' = s - 2q + \lambda_i$; $p'^k_{ij} = p_{ij}^k$. Since the case, $m = 2$ is very frequent and important in the practical field, the method of analysis for this case will be discussed here. The analysis for the other cases can be attempted in the same way without difficulty.

3. Method of Analysis

Let y_{ij} be the yield of the ith treatment in the jth block. Denoting t_i = the effect of the ith treatment, b_j = effect of the jth block, μ = grand mean and e_{ij} a random variable with zero mean and constant variance, σ^2, we take the model

$$y_{ij} = \mu + t_i + b_j + e_{ij}, \quad \begin{pmatrix} i = 1, \ldots \ldots k \\ j = 1, 2, \ldots . r \end{pmatrix}.$$

The normal equations for the design defined above are by the method of least squares,

$$T_i = \sum_j y_{ij} = (r - q)(\mu + t_i) + \sum_j b_j \tag{5}$$

$$B_j = \sum_i y_{ij} = (k - n)(\mu + b_j) + \sum_i t_i \tag{6}$$

where $\sum_j$ extends over those blocks where the ith treatment is not missing and $\sum_i$ extends over those treatments which are not missing in the jth block. For the distinction between the affected and unaffected treatments, the affected treatments will be denoted by the subscript "m" and the unaffected ones by "i".

Taking further

$$Q_i = T_i - \sum_j \bar{B}_j \tag{7}$$

$$Q_m = T_m - \sum_j{}' \bar{B}_j \tag{8}$$

where $\sum_j \bar{B}_j$ = Sum of all block means

$\sum_j{}' \bar{B}_j$ = Sum of those block means where the mth treatment is not missing

and following the method of Das (1954) the equations (5) and (6) after eliminating μ and b_j's can be reduced to

$$rt_i + q \frac{\Sigma t_m}{k-n} = Q_i \tag{9}$$

$$(r-q)\, t_m + (q-\lambda_1) \frac{\Sigma t_{m_1}}{k-n} + (q-\lambda_2) \frac{\Sigma t_{m_2}}{k-n} = Q_m \tag{10}$$

where

Σt_m = Sum over all the affected treatments;

Σt_{m_1} = Sum over those treatments which are first associates of t_m;

Σt_{m_2} = Sum over those treatments which are second associates of t_m.

Summing (10) over all the affected treatments we get

$$(r-q)\Sigma t_m + \frac{n_1(q-\lambda_1)}{k-n}\Sigma t_m + \frac{n_2(q-\lambda_2)}{k-n}\Sigma t_m = \Sigma Q_m$$

or,

$$\Sigma t_m = \frac{(k-n)\Sigma Q_m}{\nu + n_1(q-\lambda_1) + n_2(q-\lambda_2)} \tag{11}$$

where

$$\nu = (k-n)(r-q).$$

Now from equations (9) and (11) the estimate of t_i is

$$t_i = \frac{Q_i}{r} - \frac{q}{r} \cdot \frac{\Sigma Q_m}{\nu + n_1(q - \lambda_1) + n_2(q - \lambda_2)}$$

$$= \frac{Q_i}{r} - \frac{q}{r} \cdot \frac{\Sigma Q_m}{\nu + q(p - n)}. \qquad (12)$$

Again, if

ΣQ_{m_1} = Sum of Q_m's of those t_m's which are first associates of t_m; and ΣQ_{m_2} = Sum of Q_m's of those t_m's which are second associates of t_m; then summing (10) over those treatments which are first associates of t_m, we get

$$n_1(q - \lambda_1)\, t_m + \{\nu + (q - \lambda_1) p_{11}{}^1 + (q - \lambda_2) p_{12}{}^1\} \Sigma t_{m_1} + \{(q - \lambda_1) p_{11}{}^2 + (q - \lambda_2) p_{12}{}^2\} \Sigma t_{m_2} = (k - n) \Sigma Q_{m_1}. \qquad (13)$$

Since $\Sigma t_m = t_m + \Sigma t_{m_1} + \Sigma t_{m_2}$ equations (10 and 13) after eliminating Σt_{m_2} can be written as

$$A_{12} t_m + B_{12} \Sigma t_{m1} = (k - n) \left[Q_m - \frac{(q - \lambda_2)}{D} \Sigma Q_m \right] \qquad (14)$$

and

$$A_{22} t_m + B_{22} \Sigma t_{m_1} = (k - n) \left[\Sigma Q_{m_1} - \frac{C}{D} \Sigma Q_m \right] \qquad (15)$$

Solving these equations,

$$t_m = \frac{(k - n)}{A'} \left[B_{22} Q_m - B_{12} \Sigma Q_{m_1} + \frac{C B_{12} - (q - \lambda_2) B_{22}}{D} \Sigma Q_m \right] \qquad (16)$$

where

$$\left.\begin{aligned} A_{12} &= \nu - q + \lambda_2, \\ B_{12} &= (\lambda_2 - \lambda_1), \\ A_{22} &= (\lambda_2 - \lambda_1) p_{12}{}^2, \\ B_{22} &= \nu - q + \lambda_2 + (\lambda_2 - \lambda_1)(p_{11}{}^1 - p_{11}{}^2), \\ D &= \nu + q(p - n), \\ C &= (q - \lambda_1) p_{11}{}^2 + (q - \lambda_2) p_{12}{}^2, \\ A' &= A_{12} B_{22} - A_{22} B_{12}, \end{aligned}\right\} \qquad (17)$$

The sum of squares due to treatments eliminating block effects can now be obtained from $\Sigma t_i Q_i + \Sigma t_m Q_m$. The variance of different treatment contrasts are given by

$$\text{Var}\ (t_i - t_i') = \frac{2\sigma^2}{r} \tag{18}$$

$$\text{Var}\ (t_m - t_{m'}) = \frac{2(k-n)}{A'}(B_{22} + B_{12})\,\sigma^2 \tag{19}$$

where t_m and $t_{m'}$ are first associates.

$$\text{Var}\ (t_m - t_m') = \frac{2(k-n)}{A'} B_{22}\sigma^2 \tag{20}$$

where t_m and t_m' are second associates,

$$\text{Var}\ (t_i - t_m) = \sigma^2 \left[\frac{1}{r} + \frac{(k-n)}{A'}\left\{\frac{CB_{12} - (q - \lambda_2)\,B_{22}}{D}\right\} + \frac{q}{r}\cdot\frac{1}{D} + \frac{(k-n)\,B_{22}}{A'}\right]. \tag{21}$$

This type of incompleteness occurs if the number of observations missing is large. Solutions for such a design is, thus, not very much helpful in tackling the problem of missing plots. This technique can, however, be used for analysing a type of designs which can be obtained as described in the next section.

4. A Reinforced P.B.I.B. Design

From a P.B.I.B. design, a partially incomplete randomised block design with partially balanced incompleteness can always be obtained by adding $\alpha > 0$ new treatments to each block and $\beta \geqslant 0$ more blocks each containing all the treatments. We shall call this a reinforced P.B.I.B. design. In agricultural field experiments generally β is taken as zero, as the unequal block sizes lead to complication, but for experiments in bio-assay and animal husbandry such designs may have special advantages.

If a P.B.I.B. design has the parameters v', b', r', k'; n_1', n_2'; λ_1', λ_2'; $p'^{k}_{ij}\ (i, j, k = 1, 2)$, then, the parameters of the partially incomplete randomised block design with partially balanced incompleteness formed as above will be,

Total number of treatments $(k) = v' + \alpha$,

Total number of blocks $(r) = b' + \beta$, Number of affected treatments $(p) = v'$,

Number of blocks affected $(s) = b'$,

Number of treatments missing in each affected block (n) $= v' - k'$,

Number of replications missing for each affected treatment $(q) = b' - r'$

and the values of the parameters $n_1, n_2, \lambda_1, \lambda_2, p_{ij}^k$ defined in section 2 are

$$n_1 = n_1'$$

$$n_2 = n_2'$$

$$\lambda_1 = b' - 2r' + \lambda_1'$$

$$\lambda_2 = b' - 2r' + \lambda_2'$$

$$p_{ij}^k = p'^{k}_{ij}.$$

The values of the constants $A_{12}, B_{12}, A_{22}, B_{22}$, given in (17) when simplified become

$$A_{12} = r'(k' + \alpha - 1) + \lambda_2' + \beta(k' + \alpha)$$

$$B_{12} = (\lambda_2' - \lambda_1')$$

$$A_{22} = (\lambda_2' - \lambda_1') p'^{2}_{12}$$

$$B_{22} = r'(k' + \alpha - 1) + \lambda_2' + \beta(k' + \alpha) + (\lambda_2' - \lambda'_1)(p'^{1}_{11} - p'^{2}_{11}).$$

When $\beta = 0$ these coefficients can be obtained from the similar coefficients obtained by Bose and Nair (1939) replacing k' by $k' + \alpha$. As a referee of the paper Dr. K. R. Nair pointed out that when $\beta = 0$, the estimates of differences between the v' treatments and their variances both with and without recovery of inter-block information can be obtained from the corresponding expressions for the P.B.I.B. design (v', b', r', k') by replacing k' by $k' + \alpha$ in the latter and also that this is true for the general m-associate design including the B.I.B. design where $m=1$.

The average variance of differences between any two affected treatments from (19) and (20) is

$$\frac{2(k-n)}{(p-1)A'}\{(p-1)B_{22} + n_1 B_{12}\} \qquad (22)$$

Hence the efficiency factor of the comparison of any two of the v' $(= p)$ affected treatment is

$$\frac{(p-1)A'}{\nu}\left\{\frac{1}{(p-1)B_{22}+n_1B_{12}}\right\}. \tag{23}$$

To prove that the efficiency factor of the comparisons among the common set of v' treatments in reinforced P.B.I.B. design (where $\alpha > 0$, $\beta = 0$) is always greater than that of the corresponding P.B.I.B. design (with $\alpha = 0$, $\beta = 0$), it is sufficient to show that the variances of the treatment differences are decreasing function of k', as then the average variance also decreases with k' and hence the efficiency factor of the reinforced design where k' takes a greater value, *viz.*, $k' + \alpha$ becomes greater than that of the ordinary P.B.I.B. design.

Proof—

Putting

$$r'(k'-1)+\lambda_2' = R$$

i.e.,

$$k' = \frac{R+(r'-\lambda_2')}{r'} = \frac{R+c}{r'}$$

where

$$c = r' - \lambda_2'$$

in the expression of the variance

$$V = \frac{2k'(B_{22})}{B_{22}A_{12}-A_{22}B_{12}}$$

$$= 2k'\frac{[r'(k'-1)+\lambda_2'+(\lambda_2'-\lambda_1')(p'^{1}_{11}-p'^{2}_{11})]}{[\{r'(k'-1)+\lambda_2'+(\lambda_2'-\lambda_1')(p'^{1}_{11}-p'^{2}_{11})\}\{r'(k'-1)+\lambda_2'\}-(\lambda_2'-\lambda_1')^2p'^{2}_{12}]}$$

we get

$$V = \frac{2(R+c)(R+c')}{r'\{(R+c')R-M\}}$$

$$= \frac{2}{r'}\left\{\frac{R(R+c')+c(R+c')}{R(R+c')-M}\right\}$$

$$= \frac{2}{r'}\left\{1+\frac{c(R+c')+M}{R(R+c')-M}\right\}$$

$$= \frac{2}{r'}+\frac{2}{r'}\left\{\frac{c(R+c')+M}{R(R+c')-M}\right\} \tag{24}$$

where

$$c' = (\lambda_2' - \lambda_1')(p'_{11}{}^1 - p'_{11}{}^2)$$

$$M = (\lambda_2' - \lambda_1')^2 p'_{12}{}^2.$$

Now V will decrease with k' if

$$V_1 \equiv \frac{c(R + c') + M}{R(R + c') - M} \text{ decreases with } k'. \tag{25}$$

Let k' take the value $k' + \alpha$, then R becomes $R + r'\alpha$ and V_1 becomes

$$V_1' = \frac{c(R + r'\alpha + c') + M}{(R + r'\alpha)(R + r'\alpha + c') - M}. \tag{26}$$

Now $V_1 - V_1'$ on simplification reduces to

$$\frac{cr'\alpha\,[M + (R + c')(R + c' + r'\alpha)] + r'\alpha M(2R + c' + r'\alpha)}{[R(R + c') - M]\,[(R + r'\alpha)(R + r'\alpha + c') - M]}. \tag{27}$$

Since $R + c' = B_{22}$ it is always positive. Furthermore, as c, M, $[R(R + c') - M]$ and $[(R + r'\alpha)(R + r'\alpha + c') - M]$ are all positive, it is evident that (27) is a positive quantity, and hence V decreases as k' increases. Similar results can be obtained for the other variance given in (19). Hence the theorem and the aptness of using the term "reinforced" suggested to the author by K. R. Nair for naming this modified P.B.I.B. design.

5. An Example

The data analysed are taken from an agronomical trial in West Bengal which was conducted to test the relative performances of 9 different types of manures on paddy crop. The layout was originally a R.B.D. with 9 types of manures (treatments) in 9 replications. Actually the data were complete. For the sake of illustration 24 plots have been treated as missing in such a way that the missing treatments in the various blocks form a P.B.I.B. design. The Table I shows the cell frequencies of the data together with the block averages, treatment totals, both adjusted and unadjusted.

$$Q_i\,(i = 9) = -1\cdot597, \quad \sum_{m=1}^{8} Q_m = 1\cdot599.$$

TABLE I

Cell frequencies; treatment totals, both adjusted and unadjusted; block totals and block averages

(The yield figures are given in seers per plot)

Blocks	Treatments									Block totals B_j	Block averages $\bar{B}_j$
	1	2	3	4	5	6	7	8	9		
1	0	0	1	0	1	1	1	1	1	65·75 (6)	10·958
2	1	0	0	1	0	1	1	1	1	60·75 (6)	10·125
3	1	1	0	0	1	0	1	1	1	61·75 (6)	10·292
4	1	1	1	0	0	1	0	1	1	61·25 (6)	10·208
5	1	1	1	1	0	0	1	0	1	62·00 (6)	10·333
6	0	1	1	1	1	0	0	1	1	60·00 (6)	10·000
7	1	0	1	1	1	1	0	0	1	63·75 (6)	10·625
8	0	1	0	1	1	1	1	0	1	66·50 (6)	10·083
9	1	1	1	1	1	1	1	1	1	96·50 (9)	10·722
Treatment totals (T_i)	52·75 (6)	72·75 (6)	67·25 (6)	62·00 (6)	56·75 (6)	61·25 (6)	72·50 (6)	60·25 (6)	92·75 (9)	G. Total 598·25 (57)	..
Adjusted treatment totals (Q_i)	−9·555	10·111	4·403	−0·889	−6·930	−2·472	8·986	−2·055	−1·597	+·002	..

Due to rounding off errors, sum of all the Q's $= + \cdot 002$ though it should be zero. The design is a reinforced P.B.I.B. design defined in Section 4 with the parameters:

$$v' = 8, \; b' = 8, \; r' = 5, \; k' = 5, \; \alpha = 1, \; \beta = 1,$$
$$n_1' = 6, \; n_2' = 1, \; \lambda_1' = 3, \; \lambda_2' = 2,$$
$$p'_{11}{}^1 = 4, \; p'_{12}{}^1 = 1 = p'_{21}{}^1, \; p'_{22}{}^1 = 0,$$
$$p'_{11}{}^2 = 6, \; p'_{12}{}^2 = 0 = p'_{21}{}^2, \; p'_{22}{}^2 = 0.$$

In terms of the parameters of the design described in Section 2 and whose analysis is presented in Section 3 we have:

$$p = 8 \quad s = 8 \quad q = 3 \quad n = 3 \quad k = 9 \quad r = 9$$
$$n_1 = 6 \quad n_2 = 1 \quad \lambda_1 = 1 \quad \lambda_2 = 0$$
$$p_{11}{}^1 = 4 \quad p_{12}{}^1 = 1 = p_{21}{}^1 \quad p_{22}{}^1 = 0$$
$$p_{11}{}^2 = 6 \quad p_{12}{}^2 = 0 = p_{21}{}^2 \quad p_{22}{}^2 = 0$$
$$v = 36, \quad A_{12} = 33, \quad B_{12} = -1, \quad A_{22} = 0, \quad B_{22} = 35,$$
$$D = 51 \quad C = 12 \quad A' = 1155.$$

The treatment effects can be obtained from the equations (12) and (16), *i.e.*, from

$$t_i = \frac{Q_i}{9} - \frac{3}{9} \Sigma \frac{Q_m}{51} \qquad (i = 9)$$

and

$$t_m = \cdot 1818\, Q_m + \cdot 0052\, \Sigma\, Q_{m_1} - \cdot 0119\, \Sigma\, Q_m \quad (m = 1, 2, \ldots .8)$$

The solutions for t's are

$$t_1 = -1 \cdot 662 \qquad t_4 = -0 \cdot 157 \qquad t_7 = 1 \cdot 553$$
$$t_2 = 1 \cdot 788 \qquad t_5 = -1 \cdot 185 \qquad t_8 = -0 \cdot 369$$
$$t_3 = 0 \cdot 720 \qquad t_6 = -0 \cdot 500 \qquad t_9 = -0 \cdot 188$$

$$t_i (i = 9) + \sum_1^8 t_m = 0.$$

Analysis of Variance Table

Source		d.f.	S.S.	M.S.S.
Block (Unadjusted)	..	8	7·120	..
Treatment (Adjusted)	..	8	61·732	7·7165
Error	..	40	35·333	0·8833
Total	..	56	104·185	..

$$\text{Var }(t_m - t_m') = \frac{136}{385}\sigma^2 = \cdot 3120$$

when t_m and t_m' are first associates.

$$\text{Var }(t_m - t_m') = \frac{4}{11}\sigma^2 = \cdot 3212$$

when t_m and t_m' are second associates

$$\text{Var }(t_i - t_m) = \frac{1882}{6545}\sigma^2 = \cdot 2539.$$

6. Summary

A method of analysis of a R.B.D. having several missing plots which are distributed in such a way that the incompleteness in the affected blocks is partially balanced, has been obtained and illustrated by means of a numerical example. A reinforced P.B.I.B. designs with efficiency factor greater than that of the corresponding P.B.I.B. design for comparisons among the set of common treatments in the two designs has been defined, and its method of analysis is shown to follow as a special case of that of the R.B.D. with several missing plots considered earlier (Das, 1954).

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8. References

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